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Self-similar profiles in Analysis of Fluids. A 1D model and the compressible Euler equations

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Resum *(CAT)*
Presentem dos nous resultats en anàlisi de fluids relacionats amb l'existència de singularitats fent servir perfils autosimilars i anàlisi d'estabilitat al voltant d'ells. El primer resultat és una nova prova de la formació de singularitats per l'equació d'Okamoto–Sakajo–Wunsch amb petit paràmetre fent ús d'un perfil autosimilar aproximat. A la segona part trobem nous perfils autosimilars, radials i suaus, per a l'equació d'Euler compressible i isentròpica. Aquest és el primer perfil d'aquest tipus trobat pel cas de gasos monoatòmics.

Abstract *(ENG)*
We present two new results in Analysis of Fluids involving the existence of singularities via self-similar profiles and stability analysis around them. The first result is a new proof of the formation of singularities for the Okamoto–Sakajo–Wunsch equation with small parameter, which is done via a stability analysis around an approximate self-similar profile.The second result consists on the finding of new smooth radial self-similar profiles developing singularities for the isentropic compressible Euler equations. This is the first proof of such profile for the monatomic gas case.

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1. Introduction

The Navier–Stokes equation in three dimensions models the behaviour of a non-compressible fluid and is given by

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \tag{1}$$

where $x \in \mathbb{R}^3$ is physical space, the velocity of the fluid is given by the vector $u(x,t) = (u_1, u_2, u_3)$, we denote the viscosity by ν and $p(x,t)$ is the pressure. We will always denote with $u \cdot \nabla u$ the vector whose i-th component is $u \cdot \nabla u_i$. The previous equation can be interpreted in terms of Newton's second law, because the left hand-side is the convective derivative of the velocity (derivative along the trajectories of the fluid), and the right-hand side is the force exerted on the fluid. In order to close (1) (which has 4 unknowns p and u_i but only a 3-component equation), one notes that the incompressibility of the fluid implies $\mathrm{div}(u) = 0$ (volumes are preserved along trajectories), which together with (1) forms a closed system.

In order to drop the pressure one can take the curl in (1) and define the vorticity to be $\omega = \mathrm{curl}(u)$, obtaining

$$(\partial_t + u \cdot \nabla)\omega = \omega \cdot \nabla u + \nu \Delta \omega. \tag{2}$$

The incompressibility condition $\mathrm{div}\, u = 0$ ensures that u can be recovered from ω via a nonlocal operator, thanks to the Biot–Savart law:

$$u(x) = \frac{1}{4\pi}\ \mathrm{p.v.} \int_{\mathbb{R}^3} \frac{(x-y) \wedge \omega(y)}{|x-y|^3}, \tag{3}$$

where p.v. refers to the fact that the integral is done in the principal value sense[1]. The system formed by (2) and (3) has been widely studied, and the smooth existence of solutions (or a counterexample) would solve the famously known Millenium Clay Problem [14][2]. A principal difficulty of this problem is the fact that there is a nonlocal operator in the RHS, due to the fact that u is recovered from ω via a nonlocal operator. In order to reflect this quadratic nonlinear term, Constantin, Lax and Majda [8], introduced the following one dimensional model equation:

$$\omega_t = -2\omega H\omega. \tag{4}$$

Here H represents the Hilbert Transform which is a nonlocal operator in dimension 1, given by $H\omega = \frac{1}{\pi}\ \mathrm{p.v.} \int \frac{\omega(y)}{x-y}\,dy$. Some important properties of the Hilbert Transform are that it is bounded as $H\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, or that it commutes with derivatives, that is $H(f') = (Hf)'$ for all $f \in H^1(\mathbb{R})$[3]. A reference introducing the Hilbert Transform is [25].

The constant -2 in (4) makes no special role (it just rescales the solutions), and it is only fixed for the sake of comparing the model with other models. The main advantage of this model is that there are explicit formulas for the solutions, due to the following property of the Hilbert Transform:

[1] A principal value integral is done by removing a ball of radius ε around the singularity (in this case $y = x$) and taking $\varepsilon \to 0$. At infinity, one integrates in a ball $B(0,R)$ and takes $R \to +\infty$ (in the way of a Riemann indefinite integral).

[2] To be precise, the Millenium Clay Problem also requires those solutions to be finite energy, that is, $u(\cdot, t)$ uniformly bounded in $L^2(\mathbb{R})$.

[3] For the reader unfamiliar with Sobolev spaces, $H^1(\mathbb{R})$ is basically the space of $f \in L^2(\mathbb{R})$ with a derivative $f' \in L^2(\mathbb{R})$.

Theorem 1.1 (Titchmarsh theorem [26]). *Let $f, g \in L^2(\mathbb{R})$ and let $\mathbb{H} \subset \mathbb{C}$ be the upper half-plane of the complex plane. We have that $g = Hf$ if and only if there exists an holomorphic function on the upper half plane $G: \mathbb{H} \to \mathbb{C}$ such that $G(x+iy)$ converges almost everywhere to $f(x) + ig(x)$ as $y \to 0$. Moreover, in that case, G can be taken in $L^2(\mathbb{H})$, that is $\int_{\mathbb{H}} |G(z)|^2 < \infty$.*

From now on, we denote $\check{f}(x) = f(x) + iHf(x)$, which admits an holomorphic extension to the upper half plane by the previous theorem. Looking at our equation $\omega_t = -2\omega H\omega$, let G be that holomorphic extension of $\check{\omega}$. The function iG^2 is holomorphic, and its real part over the real line is $-2\omega H\omega$. Therefore, if we solve the ODE $G_t = iG^2$ for $G(\cdot, t)$ holomorphic over the upper half plane, we will recover our solution ω looking at the real part of G over the real line. The solution to that ODE is $G(z,t) = \frac{G_0(z)}{1 - iG_0(z)t}$, and the initial data $G_0(z)$ is the holomorphic extension of $\omega_0 + iH\omega_0$. One can recover ω from the real part of $G(z,t)$ for real z, and obtain the following theorem.

Theorem 1.2 (Constantin, Lax and Majda [8]). *Let ω be a solution to (4) with $\omega(\cdot, 0) = \omega_0 \in H^1(\mathbb{R})$. Then, we have that*

$$\omega(x,t) = \frac{\omega_0(x)}{(1 + H\omega_0(x)t)^2 + \omega_0(x)^2 t^2}. \tag{5}$$

In particular, it develops a singularity if and only if exists some x with $\omega_0(x) = 0$ and $H\omega_0(x) < 0$.

Example 1.3. One interesting example of initial data leading to singularity is $\omega_0(x) = \frac{x}{1+x^2}$, whose Hilbert Transform is $H\omega_0(x) = \frac{-1}{1+x^2}$ because $\frac{x-i}{1+x^2} = \frac{1}{x+i}$ is holomorphic on the upper half plane. As $H\omega_0(x) = -1$ and $\omega_0(x) = 0$, it develops a singularity. Moreover, equation (5) gives us that $\omega(x,t) = \frac{1}{1-t} F\left(\frac{x}{1-t}\right)$, for $F(x) = \frac{x}{1+x^2}$. This is called a self-similar solution, because the solution looks the same for all times t (it looks like F), just rescaled horizontally and vertically by some time-dependent factor.

2. The Okamoto–Sakajo–Wunsch model

The Okamoto–Sakajo–Wunsch model (OSW for short) was introduced in [23] as a a generalization of the CLM model. As one can observe from (4), the CLM model just substitutes the full covariant derivative $(\partial_t + u \cdot \nabla)\omega$ in 3D Euler with the term ω_t, without including a term reflecting $u \cdot \nabla\omega$. The OSW model precisely solves that, and as $u = \operatorname{curl}^{-1}(\omega)$ in 3D Euler, a reasonable choice in a 1D model is to take $u(x) = \Lambda^{-1}\omega = -\int_0^x \omega(y)\, dy$, because it is also an order -1 nonlocal operator (the $-$ sign in front is just a convention). Indeed, the OSW model reads

$$\begin{cases} \omega_t + au\omega_x = -2\omega H\omega, \\ u = \Lambda^{-1}\omega = -\int_0^x \omega, \end{cases} \tag{6}$$

where a is just some fixed parameter. Note that when we write ω_x, ω_t, we are denoting derivatives with subscripts, we will do so for the rest of this article. The case $a = 0$ recovers the CLM model. The cases $a = 2$ and $a = -2$ are also interesting, the first one is the De Gregorio model [11] and the second one was introduced by Córdoba, Córdoba and Fontelos, because its solutions yield solutions of the 2D SQG equation with some symmetries [9].

The question of the existence of singularities for this model remained a long-standing problem. First, Castro and Córdoba showed the existence of solutions developing singularities for $a < 0$ [4]. They also showed singularities in finite time for some positive $a > 0$, but using non-smooth initial data, so the question remained whether singularities can form with smooth initial data. This was recently solved by Elgindi and Jeong [13, 12], proving singularities can form for odd smooth initial data provided that $a > 0$ is small enough.

The proof is based on a $H^3(\mathbb{R})$ stability analysis around the self-similar profile for $a = 0$. One does not expect to have explicit formulas for the self-similar profile around some a small enough, but they use the self-similar profile for CLM, $F(y) = \frac{y}{1+y^2}$, and this will be very close to the exact self-similar profile provided that a is small enough. The stability analysis consists in showing that the difference between the approximate self-similar profile and the exact solution will remain much smaller than the approximate self-similar profile, so that there is a singularity dominated by this approximate self-similar profile near the blow up time.

The difference of our work with the previous work of [13, 12] is that we will do the stability analysis in a space that combines L^∞ norms for low derivatives with L^2-based norms for higher derivatives (instead of $H^3(\mathbb{R})$). In particular, this shows singularities for some non-decaying initial data (since they can be in L^∞, but not in L^2). More importantly, the L^∞ analysis requires different techniques around the singularity, and we will use the complex structure of the Hilbert Transform, which is at the core of our proof. In this short presentation, we intend to give a brief overview of the proof rather than proving all the estimates. This proof is joint work with Tristan Buckmaster, Javier Gómez-Serrano and Federico Pasqualotto.

2.1 Formulation

First of all, let us note that equation (6) gives odd solutions for odd initial data, because the Hilbert Transform changes parity (and therefore Λ^{-1} respects parity). From now on, when we talk about the OSW model, we will be talking about odd solutions.

As we already outlined, the main idea of this strategy is to consider ω solving OSW for a small enough, substract the self-similar profile $F(y) = \frac{y}{1+y^2}$ and perform some stability analysis for the difference. However, we need to choose the rescaling for our self-similar profile. For CLM we had $\frac{1}{1-t}F\left(\frac{x}{1-t}\right)$, but one expects those factors $\frac{1}{1-t}$ to change a bit as we change the parameter a.

The fundamental idea to do that temporal rescaling with the so-called modulation variables, which are fixed dynamically. This idea has been used for OSW in [12, 5] and for other equations as 2D Burgers equations with transverse viscosity [7], the Prandtl equation [6, 10], the 2D compressible Euler equation [3, 19] or the 3D compressible Euler equation [2], to cite some. The idea is to consider a rescaled profile of the form $\frac{1}{\lambda(t)}F\left(\frac{\mu(t)x}{\lambda(t)}\right)$, where $\lambda(t)$ and $\mu(t)$ are called modulation variables, and their evolutions will fixed dynamically in order to satisfy some property[4]. In our case, we will use them to fix the scalar quantities $\omega_x(0,t)$ and $H\omega(0,t)$. We will choose $\lambda(t)$ and $\mu(t)$ such that those quantities are completely absorbed by the profile, in other words, if we let $q = \omega - \frac{1}{\lambda(t)}F\left(\frac{\mu(t)x}{\lambda(t)}\right)$, we will have $q_x(0,t) = Hq(0,t) = 0$. Oddness yields $q(0,t) = Hq_x(0,t) = 0$ directly, so our way of fixing λ and μ will ensure that $\check{q}$ and $\check{q}_x$ vanish at zero, ensuring that the self-similar part is the dominating part close enough to zero.

[4]This will agree with the rescaled solutions showed for the CLM model because we will have that both $\frac{1}{\lambda(t)}$ and $\frac{\mu(t)}{\lambda(t)}$ approach $\frac{1}{1-t}$ as $a \to 0$.

As we want to show that our self-similar profile dominates, it will be easier to work in the self-similar variables, where our profile is constant and we only have to worry about the size of the difference. This is achieved by fixing $\frac{ds}{dt} = \frac{1}{\lambda(s(t))}$ (with $s(0) = 0$) and $y = \frac{\mu(s)x}{\lambda(s)}$ as the new time and space variables. Note that we make λ and μ depend on s instead of t. This will make calculations simpler, since we plan to work in (y, s) variables. If we write (6) in (y, s), the equation for q reads

$$q_s + \frac{\mu_s}{\mu} y(F_y + q_y) - \left(\frac{\lambda_s}{\lambda} + 1\right)(F + q + yF_y + yq_y) = Mq + N + a\tilde{J}. \tag{7}$$

Here, N is a quadratic term including all the terms of the form qHq and $q_y\Lambda^{-1}q$. It will not be relevant because we will work with $\|q\|_X \leqslant \varepsilon$, so $\|N\|_X \lesssim \varepsilon^2$ is much smaller (X is an appropriate Banach space that we will introduce later). The term $a\tilde{J}$ is an inhomogeneous term independent of q (only depending on F) and it will be small because a is small enough. The term Mq is a linear term in q and will be the important term that does not allow q to grow a lot, yielding stability.

2.2 The modulation

In order to control (7), we need to control the modulation variables $\mu(s)$ and $\lambda(s)$. As we already told, we are going to fix λ and μ such that $q'(0) = Hq(0) = 0$, which gives us a second order cancellation of $\check{q}$ around the origin. Therefore, we will assume that the initial data satisfy $q_0'(0) = 0$, $Hq_0(0) = 0$ and we will get a system of evolution equations for (λ, μ) just by taking a spatial derivative or a Hilbert Transform in (9) and asking for $\frac{d}{ds} q_y(0) = 0$ and $\frac{d}{ds} Hq(0) = 0$. Doing so, we get

$$\begin{cases} \left(\frac{\lambda_s}{\lambda} + 1 - ak_2\right) = aH(F_y\Lambda^{-1}q + q_y\Lambda^{-1}F + q_y\Lambda^{-1}q)(0), \\ \left(\frac{\mu_s}{\mu} - ak_1\right) = 2aH(F_y\Lambda^{-1}q + q_y\Lambda^{-1}F + q_y\Lambda^{-1}q)(0), \end{cases} \tag{8}$$

where $k_2 = H(F_y\Lambda^{-1}F)(0) = \log(2) - 1/2$ and $k_1 = 2k_2 - 1 = -2 + 2\log(2)$. Therefore, in order to control the modulation, we just need to control the quantity $H(F_y\Lambda^{-1}q + q_y\Lambda^{-1}F + q_y\Lambda^{-1}q)(0)$, which seems feasible, because we will take q to be small enough.

Using (8), it is useful to rewrite equation (7) as

$$q_s = M_0q + aM_1q + aP + N + aJ, \tag{9}$$

where

$$\begin{aligned} aP &= (F + q - yF_y - yq_y)H(q_y\Lambda^{-1}F + F_y\Lambda^{-1}q + q_y\Lambda^{-1}q)(0), \\ M_0q &= -q - yq_y - 2qHF - 2FHq. \end{aligned}$$

The term $(M_0q + aM_1q)$ is the linear term from before, but we have put together all the terms $O(a)$ in aM_1q, and keep the important terms in M_0q. The term aP comes from the modulation we have just discussed. Lastly, N is a non-linear term as before, and aJ is the inhomogeneous term different from the one before. One also has that $\check{J}$ and $\check{J}_y$ are zero at the origin.

2.3 Bootstrap argument

The whole argument consists in showing that q remains small in a suitable Banach space under (9). We will assume the initial bounds

$$\|\psi\check{q}_0\|_{L^\infty} \leqslant \varepsilon_1/2, \quad \|\psi\partial_{yy}\check{q}_0\|_{L^\infty} \leqslant \varepsilon_2/2, \quad \|y\partial_y q_0\|_{L^2} \leqslant \varepsilon_3/2 \quad \text{and} \quad \|\sqrt{1+y^2}\partial_y^4 q_0\|_{L^2} \leqslant \varepsilon_4/2, \tag{10}$$

for some $a \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1$ and $\psi = \frac{1+y^2}{y^2}$. Then, we will try to show the uniform bounds

$$\|\check{q}(\cdot,s)\|_{L^\infty} \leqslant \varepsilon_1, \quad \|\check{q}_{yy}(\cdot,s)\|_{L^\infty} \leqslant \varepsilon_2, \quad \|yq_y(\cdot,s)\|_{L^2} \leqslant \varepsilon_3 \quad \text{and} \quad \|\sqrt{1+y^2}\partial_y^4 q(\cdot,s)\|_{L^2} \leqslant \varepsilon_4. \tag{11}$$

The core of the proof is the following proposition.

Proposition 2.1. *Suppose that q_0 satisfies (10) and let q be the solution to (9). Then, the bounds (11) hold uniformly in time s.*

Let $\|\cdot\|_X$ be the norm consisting on adding up all the norms in (10), and the Banach space X formed by functions for which that norm is finite. The norm coming from adding up all the norms in (11) may seem different (because of the ψ weight in L^∞) but is in fact an equivalent norm if we restrict to the space of q with $\check{q}(0) = 0$ and $\check{q}_y(0) = 0$.

The strategy for proving Proposition 2.1 will be a bootstrap argument, which we are going to describe. The local wellposedness theory (which follows from the general theory of Kato and Lai [18]), gives us that the previous norms evolve continuously with respect to the time s. Therefore, in order to show (11) globally in time, we argue by contradiction, assuming that (11) are satisfied up to some time s_0, and that one of the bounds is broken at time s_0. If we get a contradiction from that, we will have that (11) are global in time. Therefore, for each of the bounds, we just suppose that one is the first to be broken and we arrive to a contradiction.

The reasons behind why do we need such variety of norms are mainly the following. First of all, we need to control $\check{q}$ and not just q due to the fact that we will work with the complexified equation for $\check{q} = q + iHq$, as in CLM. Secondly, we also need their second derivatives to get some nontrivial control at the origin (remember $\check{q}(0) = \check{q}_y(0) = 0$). Also, the L^2 norms at higher derivatives avoid a loss of derivative that we face for the estimates on $\|\partial_y^2\check{q}\|_\infty$. Another good property about our norm is that one can derive the inequality

$$\|H(F_y\Lambda^{-1}q + q_y\Lambda^{-1}F + q_y\Lambda^{-1}q)\|_{L^\infty} \leqslant C\|q\|_X, \tag{12}$$

for some constant $C > 0$, which allows to control the modulation.

2.4 Example of a simplified equation

Let us illustrate the proof of Proposition 2.1 with a much simpler example. We have that aM_1q and aP will be small because M_1q, P are bounded and a is small enough. On the other hand, N will be small because it is nonlinear, so a term like qHq will be simply bounded by ε_1^2. Therefore, let us assume we just have to deal with the much simpler equation $q_s = M_0q + aJ$, instead of (9). We will show how the bootstrap works for the $\|\check{q}\|_{L^\infty}$ estimates in this equation.

Lemma 2.2. *Let us assume that q satisfies $q_s = M_0 q + aJ$ and the initial data $q(\cdot, 0) = q_0(\cdot)$ satisfies assumptions (10). Then, we have the estimate $\|\check{q}(\cdot, s)\|_{L^\infty} \leqslant \varepsilon_1$ holds for all times $s \geqslant 0$.*

Proof. The point here is that $M_0 q$ admits a nice formula for its complexification, namely $\widetilde{M_0 q} = -\check{q} - y\check{q} + 2i\check{F}\check{q}$[5]. Using the characteristic curves $y = \eta_{y_0}(s) = y_0 e^s$ in our equation $\check{q}_s = \widetilde{M_0 q} + a\check{J}$, we obtain that $\beta_{y_0}(s) = \check{q}(\eta_{y_0}(s), s)$ satisfies

$$\frac{\partial}{\partial s}\beta_{y_0}(s) = -\beta_{y_0} + 2i\check{F}(\eta_{y_0}(s))\beta_{y_0}(s) + a\check{J}(\beta_{y_0}(s)).$$

The problem with this equation is that $2i\check{F}(0) = 2$, which spoils the damping coming from $-\beta_{y_0}$. As $\check{F}$ decays, there is no problem for big enough $y = \eta_{y_0}(s)$. To solve the issue near zero we introduce the weight $\psi = \frac{1+y^2}{y^2}$, which decreases quadratically along trajectory lines near the origin, giving an extra -2 damping in that region. We obtain

$$\frac{\partial}{\partial s}(\beta_{y_0}(s)\psi(y)) = \left[-3 + 2i\check{F}(y) + \frac{2}{\psi(y)}\right]\beta_{y_0}(s)\psi(y) + a\check{J}(y)\psi(y),$$

where the $+\frac{2}{\psi(y)}$ in the brackets comes from the derivative hitting the numerator of $\psi(y)$. The point here is that the term in brackets is uniformly negative. For small y we have $|2i\check{F}(y)| \leqslant 2$ and $\frac{2}{\psi(y)}$ is very small, while for high y the opposite happens. Doing the calculation for all intermediate values of y one concludes that the term in brackets is uniformly negative (for example, smaller than $-1/10$). Note also that $|a\check{J}(y)\psi(y)| \leqslant Ca$ for some absolute constant C because $|\check{J}| \lesssim y^2$ near the origin (it has a second order zero at the origin). Therefore, letting $\Omega(s) = \beta_{y_0}(s)\psi(\eta_{y_0}(s)) = \check{q}(y, s)\psi(y)$, we conclude

$$\frac{\partial}{\partial s}\Omega = f(s)\Omega(s) + \mathcal{E}, \tag{13}$$

with some error $|\mathcal{E}| \leqslant Ca$ and with $f(s) \leqslant \frac{-1}{10}$. We also have that $\Omega(0) \leqslant \frac{\varepsilon_1}{2}$ from our assumption (10).

Finally, choose $a \ll \varepsilon_1$ small enough and assume that $|\Omega(s)| \leqslant \varepsilon_1$ is broken at time s_0, so we have $\Omega(s_0) = \varepsilon_1$ and $\Omega'(s_0) \geqslant 0$[6]. Those assumptions directly give a contradiction from (13) at $s = s_0$, provided that $0 > -\frac{1}{10}\varepsilon_1 + Ca$, which can be ensured, because $a \ll \varepsilon_1$.

Finally, as $|\Omega(s)| \leqslant \varepsilon_1$, we obtain that $\|\check{q}(\cdot, s)\psi(\cdot)\|_{L^\infty} \leqslant \varepsilon_1$ and therefore $\|\check{q}(\cdot, s)\|_{L^\infty} \leqslant \varepsilon_1$. $\square$

2.5 Conclusion

Using Proposition 2.1, one concludes that $\|q\|_X$ remains uniformly bounded, provided q_0 satisfies (10). For now on, let us fix those assumptions for q_0. Therefore, we have that $\omega = F + q$ solves the original OSW equation in the rescaled variables. Moreover, we know that $s = +\infty$ corresponds to some finite time $t = T^*$ previous to the rescaling because $\frac{dt}{ds} = \lambda(s)$ and λ decays exponentially as one can see from (8), (12) and the fact that a is small enough.

[5] This should not be surprising, as no term in $M_0 q$ has an a in front, so all the terms come from CLM, which admits a solution via complexification.

[6] The same reasoning applies for $\Omega(s_0) = -\varepsilon_1$ and $\Omega'(s_0) \leqslant 0$.

Theorem 2.3. *There exist some values a_0, ε_i, δ, which can be chosen all small enough such that the following holds. Let $a \in (0, a_0]$ and $\omega_0(x) = F(x) + q_0(x)$ all odd, where q_0 satisfies the bounds (10). Then, we have that the solution to OSW with initial data ω_0 is given by*

$$\omega(x,t) = \frac{1}{\lambda(t)}\left(F\left(\frac{\mu(t)x}{\lambda(t)}\right) + q\left(\frac{\mu(t)x}{\lambda(t)}, s(t)\right)\right),$$

where q satisfies the global bounds (11) and $\lambda(t) \to 0$ as $t \to T^*$ [7]. In particular, we have that both $H\omega(0,t)$ and $\omega_x(0,t)$ blow up as $t \to T^*$.

Remark. The assumptions on q_0 can be formulated in terms of ω_0. Basically we require that ω_0 is odd and close enough to the self-similar profile F. The assumption $Hq(0) = q_y(0) = 0$ can be satisfied by modifying the modulation variables initial conditions $\lambda(0)$, $\mu(0)$, so it does not impose a restriction on ω_0.

Remark. As a final comment, let us note that our Theorem includes self-similar singularities for compactly supported smooth initial data. As the self-similar profile F decreases, one can choose q_0 so that $q_0(x) = -F(x)$ for $x \notin [-R, R]$ and R big enough, while still satisfying assumptions (10). Choosing q_0 smooth also ensures that $F + q_0$ is smooth, so we can construct compactly supported and smooth initial data ω_0 so that OSW blows up in a self-similar way.

3. Self-similar profiles for isentropic compressible Euler

The other result of this article is joint work with Tristan Buckmaster and Javier Gómez-Serrano, and focuses on proving the existence of radial smooth self-similar profiles for the isentropic compressible Euler or Navier–Stokes. The equation is given by

$$\begin{cases} \rho\partial_t u + \rho u \nabla u = -\nabla p + \nu \Delta u, \\ \rho_t + \operatorname{div}(u\rho) = 0, \\ p(\rho) = \dfrac{1}{\gamma}\rho^\gamma, \end{cases} \tag{14}$$

where the viscosity $\nu = 0$ corresponds to Euler and $\nu > 0$ to Navier–Stokes. The first equation corresponds to the conservation of momentum (or equivalently, Newton's second law). The second equation is the conservation of mass along trajectory lines and the third one is the isentropic law for the pressure of an ideal gas. The parameter γ is called adiabatic constant and we will center here in the monatomic ideal gas case which corresponds to $\gamma = 5/3$. We will also restrict to the Euler case, that is $\nu = 0$. In the forthcoming paper [1], we extend the range of γ, we obtain other types of profiles and we use the Euler profiles to perform a stability analysis for the Navier–Stokes case, obtaining self-similar singularities for the isentropic compressible Navier–Stokes equations.

This approach is inspired in the recent series of papers [20, 22, 21], which solved the outstanding problem of the existence of singularities for compressible Navier–Stokes with smooth initial data, for some values of γ. However, no smooth self-similar profiles were found for the monatomic gas ($\gamma = \frac{5}{3}$) before

[7] We are writing $\lambda(t)$ and $\mu(t)$ for notational simplicity instead of the more correct alternatives $\lambda(s(t))$ and $\mu(s(t))$.

this work, so our result is completely new. The type of profiles found is also new, and in particular they concentrate faster than those found in [20].

3.1 Self similar equation

First of all, we will always work with radial solutions to (14). We also impose $\nu = 0$, as we will work with the Euler equations. We write the equation in radial coordinates $u(R)$ and $\rho(R)$, with u being a vector in the radial direction and $R = |x|$. Then, we define $\alpha = \frac{\gamma-1}{2}$ (which is $\frac{1}{3}$ for $\gamma = \frac{5}{3}$) and perform the change of variables $w(R,t) = u(R,t) + \frac{1}{\alpha}\rho^\alpha$ and $z(R,t) = u(R,t) - \frac{1}{\alpha}\rho^\alpha$. This change of variables was proposed by Riemann [24] and thus (w,z) are usually called the Riemann invariants. The equation obtained for the Riemann invariants is

$$\begin{cases} \partial_t w + \left(\dfrac{1+\alpha}{2}w + \dfrac{1-\alpha}{2}z\right)\partial_R w + \dfrac{\alpha}{2R}(w^2 - z^2) = 0, \\ \partial_t z + \left(\dfrac{1-\alpha}{2}w + \dfrac{1+\alpha}{2}z\right)\partial_R w - \dfrac{\alpha}{2R}(w^2 - z^2) = 0, \end{cases} \tag{15}$$

which exhibit a lot of symmetry. We are looking for self-similar solutions, and inspired by the two parameter family of scaling symmetries of (15), we try the ansatz $w(R,t) = \frac{1}{r}\frac{R}{T-t}W\big(\frac{R}{(T-t)^{1/r}}\big)$ (and the same for z). Here, r is just a parameter, so the factor $\frac{1}{r}$ is just a constant factor that will make computations simpler. We also observe that the lower the r, the faster that the profile will expand, as the exponent $\frac{1}{r}$ will be bigger. Defining $\xi = \frac{R}{(T-t)^{1/r}}$, one obtains that the self-similar profiles $W(\xi)$, $Z(\xi)$ satisfy

$$\begin{cases} \xi\partial_\xi W = \dfrac{-rW - (1+2\alpha)W^2/2 + (1-\alpha)WZ/2 - \alpha Z^2/2}{1 + (1+\alpha)W/2 + (1-\alpha)Z/2} = \dfrac{N_W(W,Z)}{D_W(W,Z)}, \\ \xi\partial_\xi Z = \dfrac{-rZ - (1+2\alpha)Z^2/2 + (1-\alpha)WZ/2 - \alpha W^2/2}{1 + (1-\alpha)W/2 + (1+\alpha)Z/2} = \dfrac{N_Z(W,Z)}{D_Z(W,Z)}, \end{cases} \tag{16}$$

which is an algebraic autonomous dynamical systems (the change $\tilde{\xi} = \log(\xi)$ makes the system autonomous) in two dimensions. Note that this dynamical system depends on two parameters γ and r, and that N_W, N_Z, D_W, D_Z are just polynomials in W and Z.

This formulation has been known since Guderley [16], and it is not difficult to prove the existence of self-similar profiles of limited regularity C^k from those equations. However, the question of existence of smooth solutions to (16) is much more difficult. We have plotted the phase portrait in Figure 3.1. The fundamental problem for the existence of smooth profiles is the singular point P_2, defined as the intersection of $D_Z = 0$ and $N_Z = 0$. The solutions we are looking for start at point P_1 (point at infinity in the asymptotic direction $(1,-1)$) at $\xi = 0$, then pass smoothly through P_2 at $\xi = 1$ and end up reaching $P_3 = (0,0)$ at $\xi = +\infty$. The profiles starting at P_1 and ending at P_3 ensure that the solution $w(R,t) = \frac{R}{T-t}W(\xi)$ is non-zero at $R = 0$ and does not grow as $R \to +\infty$. Even though the profile is singular at $\xi = 0$, note that the real solution $w(R,t)$ before rescaling is not, due to the factor R multiplying[8].

[8] If one prefers profiles not singular at 0, one can equivalently work with $\xi W(\xi)$ as a profile and modify the rescaling appropriately.

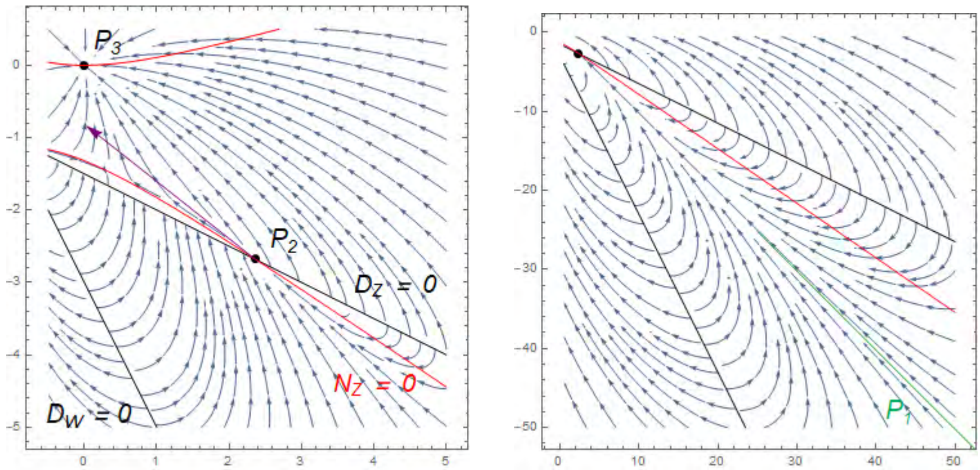

Figure 3.1: Phase portrait of (16) at $r = 1.1$ and $\gamma = 5/3$ at different scales.

As one can see in the phase portrait, solutions going from P_1 to P_3 have to pass through the line $D_Z = 0$, which is a major problem in terms of the equation (16), as it corresponds to $\xi\partial_\xi Z$ blowing up. Thus, the only possibility for a solution crossing $D_Z = 0$ smoothly, is to cross through the point P_2, where both D_Z and N_Z cancel and the quotient may remain bounded.

3.2 Local analysis around P_2

One can look at the modified dynamical system $W_\zeta = N_W D_Z$ and $Z_\zeta = N_Z D_W$, which just corresponds to an appropriate reparametrization in time of our original system (because (W_ξ, Z_ξ) is proportional to (W_ζ, Z_ζ), so trajectories are locally preserved). The point P_2 is an equilibrium point of the new system and we can perform a local analysis of this point looking at the eigenvalues of the Jacobian of $(N_W D_Z, N_Z D_W)$.

Doing that, we see there exists a value $r^*(\gamma) > 1$ (which is $3 - \sqrt{3}$ for $\gamma = \frac{5}{3}$) such that both eigenvalues of the Jacobian are positive for $r \in [1, r^*(\gamma))$. Moreover, letting $k(r) \geqslant 1$ to be the quotient of the eigenvalues, we have that $k(r)$ is monotonically increasing from $k(1) = 1$ to $\lim_{r\to r^*} k(r) = +\infty$. The dynamical system theory tells us that P_2 is a focus, and when $k \notin \mathbb{N}$, generic solutions near P_2 will have limited C^k regularity as we tend to P_2. However, there will be two exceptional invariant curves of the system passing smoothly through P_2, which correspond to two smooth solutions of (16) through P_2. One of them agrees up to order $\lfloor k \rfloor$ with all the non-smooth trajectories, and we will focus on that one (which we simply refer as "the" smooth solution).

Taking derivatives of our ODE (16) and evaluating at P_2, one can find a recurrence for the Taylor coefficients (W_n, Z_n) of the smooth solution. We can see that they define a convergent series continuous with respect to the parameter r. More importantly, we get an expression for Z_n of the form $Z_n = \frac{1}{n-k(r)}\check{Z}_n$, where $\check{Z}_n$ is some polynomial expression in the previous coefficients. This is of paramount importance, because it indicates that the n-th Taylor coefficient blows up as $k(r) \to n$.

3.3 Statement and idea of the proof

Theorem 3.1 (Buckmaster, Cao-Labora and Gómez-Serrano [1]). *Let $\gamma = 5/3$. There exists a value of r for which $k(r) \in (3, 4)$, such that there is a smooth solution to (16) emanating from the point P_1 at $\xi = 0$, passing smoothly through P_2 at $\xi = 1$ and reaching P_3 asymptotically as $\xi \to +\infty$.*

First of all, the existence of a solution emanating from P_1 follows easily from calculating its Taylor series recurrence and showing convergence. The rest of the proof is divided in the two following propositions.

Proposition 3.2. *Let $\gamma = 5/3$. There exists a value of r with $k(r) \in (3,4)$ such that the smooth solution around P_2 coincides with the unique solution emanating from P_1.*

This proposition is proved via a shooting argument in r. We prove that for $k(r) = 3+\varepsilon$ and $k(r) = 4-\varepsilon$ (for ε small enough) the solution around P_2 stays respectively below and above the solution emanating from P_1. Continuity with respect to r ensures there is an intermediate value of r for which both solutions agree. In order to show the expected behaviour of the solution for $k = 3+\varepsilon$ and $k = 4-\varepsilon$ we take advantage of the fact that $|Z_3|$ and $|Z_4|$ blow up, respectively, as $\varepsilon \to 0$. The argument is formalised via a concatenation of barrier arguments involving detailed local behaviour of the solution around P_2.

Proposition 3.3. *Let $\gamma = 5/3$ and $k \in (3,4)$. We have that the smooth solution at P_2 reaches point P_3 at $\xi = +\infty$.*

This proposition is also shown via barrier arguments. We also need to take into account the blow-up of the Taylor series as k is close to 3 or 4, however we need to prove the result for all the intermediate values as well. The strategy is to introduce a reparametrization in the barrier (singular as k approaches natural numbers) that desingularizes the barrier conditions. After desingularization, the barriers are proved via computer-assisted proofs. A recent example in using computer assisted-proofs for proving barrier arguments is [17], where self-similar profiles for a model of polytropic gaseous stars are obtained. A general survey of computer-assisted proofs in PDEs is [15].

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References

[1] T. Buckmaster, G. Cao-Labora, J. Gómez-Serrano, "Smooth imploding solutions for 3D compressible fluids", Forthcoming work (2021).

[2] T. Buckmaster, S. Shkoller, V. Vicol, "Formation of point shocks for 3D compressible Eu-

ler", Preprint (2019), https://arxiv.org/abs/1912.04429.

[3] T. Buckmaster, S. Shkoller, V. Vicol, "Formation of shocks for 2D isentropic compressible Euler", Preprint (2019), https://arxiv.org/abs/1907.03784.

[4] Á. Castro, D. Córdoba, "Infinite energy solutions of the surface quasi-geostrophic equation", *Adv. Math.* **225**(4) (2010), 1820–1829.

[5] J. Chen, T.Y. Hou, D. Huang, "On the finite time blowup of the De Gregorio model for the 3D Euler equation", Preprint (2019), https://arxiv.org/abs/1905.06387.

[6] C, Collot, T.-E. Ghoul, S. Ibrahim, N. Masmoudi, "On singularity formation for the two dimensional unsteady Prandtl's system", Preprint (2018), https://arxiv.org/abs/1808.05967.

[7] C. Collot, T.-E. Ghoul, N. Masmoudi, "Singularity formation for Burgers equation with transverse viscosity", Preprint (2018), https://arxiv.org/abs/1803.07826.

[8] P. Constantin, P.D. Lax, A. Majda, "A simple one-dimensional model for the three-dimensional vorticity equation", *Commun. Pure Appl. Math.* **38**(6) (1985), 715–724.

[9] A. Córdoba, D. Córdoba, M.A. Fontelos, "Formation of singularities for a transport equation with nonlocal velocity", *Ann. of Math.* **162** (2005), 1377–1389.

[10] A.-L. Dalibard, N. Masmoudi, "Separation for the stationary Prandtl equation", *Publ. Math. Inst. Hautes Études Sci.* **130**(1) (2019), 187–297.

[11] S. De Gregorio, "On a one-dimensional model for the three-dimensional vorticity equation", *J. Stat. Phys.* **59**(5-6) (1990), 1251–1263.

[12] T.M. Elgindi, T.-E. Ghoul, N. Masmoudi, "Stable self-similar blowup for a family of nonlocal transport equations", Preprint (2019), https://arxiv.org/abs/1906.05811.

[13] T.M. Elgindi, I.-J. Jeong, "On the effects of advection and vortex stretching", *Arch. Ration. Mech. Anal.* **235**(3) (2020), 1763–1817.

[14] C.L. Fefferman, "Existence and smoothness of the Navier–Stokes equation", *The millennium prize problems* **57** (2006), 67.

[15] J. Gómez-Serrano, "Computer-assisted proofs in PDE: a survey", *SeMA J.* **76**(3) (2019), 459–484.

[16] K.G. Guderley, "Starke kugelige und zylinrische verdichtungsstosse in der nahe des kugelmitterpunktes bzw. Der zylinderachse", *Luftfahrtforschung* **19** (1942), 302.

[17] Y. Guo, M. Hadzic, J. Jang, M. Schrecker, "Gravitational collapse for polytropic gaseous stars: self-similar solutions", Preprint (2021), https://arxiv.org/abs/2107.12056.

[18] T. Kato, C.Y. Lai, "Nonlinear evolution equations and the Euler flow", *J. Funct. Anal.* **56**(1) (1984), 15–28.

[19] J. Luk, J. Speck, "Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity", *Invent. Math.* **214**(1) (2018), 1–169.

[20] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel, "On smooth self similar solutions to the compressible Euler equations", Preprint (2019), https://arxiv.org/abs/1912.10998.

[21] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel, "On the implosion of a three dimensional compressible fluid", Preprint (2019), https://arxiv.org/abs/1912.11009.

[22] F. Merle, P. Raphaël, I. Rodnianski, J. Szeftel, "On blow up for the energy super critical defocusing nonlinear Schrödinger equations", *Invent. Math.* (2021), 1–167.

[23] H. Okamoto, T. Sakajo, M. Wunsch, "On a generalization of the Constantin–Lax–Majda equation", *Nonlinearity* **21**(10) (2008), 2447.

[24] B. Riemann, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Verlag der Dieterichschen Buchhandlung, 1860.

[25] T. Tao, "Lecture notes 4 for 247A", `https://www.math.ucla.edu/~tao/247a.1.06f/`.

[26] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, 1948.